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Weinberg like sum rules revisited

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Abstract

The generalized Weinberg sum rules containing the difference of isovector vector and axial-vector spectral functions saturated by both finite and infinite number of narrow resonances are considered. We give a historical survey and summarize the status of these sum rules analyzing their overall agreement with phenomenological Lagrangians, low-energy relations, parity doubling, hadron string models, and experimental data.

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1. Introduction

The sum rules equating certain moments of the spectral weight functions of two-point current correlators turned out to be an extremely fruitful concept in the hadron physics. They were first proposed by Weinberg who considered the difference of vector (V) and axial-vector (A) correlators [1], later this difference was recognized to be an order parameter of spontaneous chiral symmetry breaking in QCD in the chiral limit. The original derivations of such sum rules suffered from a lack of rigour as long as they were based on *ad hoc* postulates about the high-energy behaviour of certain combinations of two-point correlators and about the possible nature of Schwinger terms of the current commutators. The original Weinberg's derivation rested on a proof of equality of V and A Schwinger terms and on the assumption of asymptotic chiral $SU(2) \times SU(2)$ symmetry at high momentum, thus providing for the first time a concrete realization of the notion of chiral symmetry at short distances. The latter proposal was immediately generalized in [2], where it was shown that the idea of asymptotic symmetries may serve as a powerful tool for deriving various interesting results in the pole approximation. These papers were followed by the $SU(3) \times SU(3)$ generalization of Weinberg sum rules [3] and subsequent early applications to hadron physics [4-8]. The Weinberg sum rules were exploited for derivation of electromagnetic mass difference [9] and Das-Mathur-Okubo sum rule [10], both results have been widely used up to now. The former one relies essentially on the validity of the second Weinberg sum rule, otherwise this mass difference is not finite (later it was demonstrated [11] that the use of the second spectral-function sum rule is not necessary if an exchange by intermediate weak gauge boson is taken into account). This and other phenomenological observations called for a rigorous justification of spectral-function sum rules. An important requirement was also the universality of such a justification, as, say, the original method used in [1] to derive the equality of V and A Schwinger terms does not work, generally speaking, in the scalar case.

The first attempt in this direction was the proposal to replace the current algebra by the algebra of gauge fields [12], the Schwinger terms in the latter are explicitly known c-numbers. Considerable progress in understanding the Weinberg sum rules was achieved due to the introduction of Wilson's Operator Product Expansion (OPE) at short distances [13] as a substitute of Lagrangian models, the new techniques was applied to the sum rules already in the original paper [13]. The Wilson's proposal happened to be very useful tool for analysis of convergence of spectral-function sum rules. Wil-son proved, for instance, that if his OPE method is true, the necessary and sufficient condition for the validity of Weinberg sum rules is that the chiral $SU(2) \times SU(2)$ group be an exact invariance (see also [14] for a more general proof). The discovery of asymptotic freedom in non-Abelian gauge theories [15,16] inspired to consider the status of spectral-function sum rules within the framework of asymptotically free theories [17], in particular, exploiting the techniques of Wilson's OPE [18]. A general recipe for extracting exact spectral-function sum rules for asymptotically free theories was presented in [19].

Weisberger showed [20] that the Wilson's results can be reproduced in Lagrangian theory by means of the use of the renormalization-group equation. In particular, the general theoretical criterium for the validity of spectral sum rules was derived: the symmetry-breaking terms in a renormalizable Lagrangian must have canonical dimensions $\delta \leq 3$. Such a soft symmetry breaking can be implemented in practice by mass terms and by scalar fields with nonvanishing vacuum expectation value. Provided this condition is satisfied, the propagators approach their symmetric values in the asymptotic spacelike limit. Choosing then certain linear combinations of propagators whose asymptotics will be less singular than that of the individual terms and applying the spectral representations for those combinations, one obtains precisely the Weinberg sum rules. To realize the program completely, one needs to know nonleading asymptotic terms which depend on the symmetry breaking effects. The earlier studies of chiral symmetry breaking within various spectral-function sum rules can be found in [2,17,18,21-25].

In summary, it became clear that the two Weinberg sum rules are very general and do not depend on the dynamics of chiral symmetry breaking in the vacuum for the asymptotically free theories, such as QCD, while the higher-order sum rules do depend on that dynamics, hence, on the specific details of the QCD Lagrangian.

The invention of famous ITEP sum rules [26] was a significant progress in development of OPE-based sum rules. Within that method, one improves the convergence and suppresses the contribution of higher excitations by performing the Borel transformation in Euclidean space, and then one parametrizes all non-perturbative effects by means of a few condensates entering the numerators of OPE. As a result, the perturbative and non-perturbative contributions become effectively factorized, which permits to make numerous predictions having at hand only several inputs – the phenomenological values of condensates. There is still no complete understanding why this method works so well in the phenomenology. In particular, the application of the same method to a solvable quantum-mechanical problem [27] shows that when the hadron continuum is not known and is modelled by an effective continuum threshold (this very situation one has usually in practice), the systematic uncertainties of the method cannot be controlled. The assumption of dominance of the lowest-lying resonances is sometimes also in doubt [28]. A possibility to take into account the higher excitations appeared within the Finite Energy Sum Rules (FESR) [29-31] (see, e.g., [32] for references), which represent a kind of extension of the Weinberg sum rules based on analytic properties of correlation functions and on quark-hadron duality.

About twenty years ago it was realized that the sum rules can be directly confronted with experiment through semi-leptonic τ-lepton decays, namely the V and A spectral functions can be reconstructed in the kinematical range limited by the τ-lepton mass. The first attempt was undertaken in [33] using the ARGUS data [34]. This analysis was followed by improved versions [35-37]. Subsequently, much more precise data of the ALEPH and OPAL collaborations on the V and A spectral functions [38,39] gave rise to a large series of papers devoted to the extraction of hadronic parameters, such as condensates, with the help of FESR and other methods [40-52].

In the last decade the sum rules saturated by narrow resonances have found numerous applications in the phenomenology, it would require a long paper to survey this activity. In this respect, a question might even appear whether new papers on such well known sum rules are really needed. We believe, however, that some new trends in the phenomenology invite to return to foundations of resonance sum rules and to revise them. Each scheme of resonance saturation implies a certain pattern for the chiral symmetry breaking at low energies, the most known pattern is based on the conception that the a_1-meson is the chiral partner of the ρ-meson and the σ-meson is that of pion. If the Wigner-Weyl realization of chiral symmetry was somehow restored maintaining confinement in QCD, these chiral partners would be degenerate. The same pattern is used for construction of many effective quark models, equal resonance content provides a possibility to match these models to QCD sum rules establishing thereby a correspondence of effective models to the fundamental theory (see, e.g., [53-67] and references therein). There is, however, an alternative possibility [68] where the ρ-meson is a "would-be" chiral partner of pion. It is not excluded that this pattern would be preserved even if the chiral symmetry was restored – the so-called vector manifestation scenario [69]. To a certain extent, the recent phenomenological observations yield an unexpected support for this scenario – the ρ-meson belongs to the leading Regge trajectory, the states lying on such trajectories, probably, do not have parity partners [70-72] and if the chiral symmetry is effectively restored above the chiral symmetry breaking scale, the chiral partner of the a_1-meson seems to be the $\rho(1450)$-resonance, the first "radial" excitation of the ρ-meson (such a possibility was explored in [67]). This example shows that further systematic studies of both saturation schemes and relations between the QCD sum rules and phenomenological Lagrangians are needed, we will address to these subjects.

Recently the resonance sum rules were employed to demonstrate that the chiral symmetry is realized in the Wigner-Weil mode in the upper part of meson spectrum [73-75], the first attempt of this kind seems to go back to [76], where the baryon sector was analyzed. These attempts boiled down to justification of parity doubling among the highly excited states, the procedure turned out to be model-dependent, thus not replying unambiguously whether the chiral symmetry is restored or not. Another approach was put forward in [77], where the spectrum was split into the "chirally symmetric" and "nonsymmetric" parts. The first part, after summation over resonances and comparison with the OPE, yields no contribution to the condensates responsible for the chiral symmetry breaking. If the chiral symmetry gets restored, the second part has to represent the asymptotically vanishing corrections to the first part. Technically, one should fix an ansatz for mass spectrum, e.g. take the linear one, calculate its input parameters from the imposed constraints, and compare with the experimental data and known theoretical relations to check whether this works. Unfortunately, the existing uncertainties both in the experimental data and in the OPE condensates do not permit to perform this reliably.

The chiral symmetry restoration still remains a rather iffy concept [72,78] in spite of all efforts to justify it [79], for this reason we would prefer to use the term "parity doubling", the latter is easier to compare with the actual spectroscopy. As parity doubling seems to be an important observable phenomenon [80], it is interesting to consider to what extent the approximate parity degeneracy can take place in the sum rules saturated by finite number of resonances (leaving aside the case of trivial degeneracy) [81] and check, if possible, whether such mass spectra are more preferable from the phenomenological point of view in comparison with mass spectra without approximate parity doubling. This subject will be also addressed in the present work.

The rest of review is organized as follows. In Sect. 2 we concern a relation between the generalized Weinberg sum rules and the phenomenological Lagrangians of effective field theory. Sect. 3 deals with the same sum rules in the correlator approach. Sect. 4 is devoted to solutions of sum rule equations at different saturation schemes and additional assumptions, with the main emphasis being placed on the possibility for parity doubling. In Sect. 5 we comment on some problems emerging in the sum rules with infinite number of states. Our conclusions are summarized in Sect. 6.

2. Sum rules: Lagrangian approach

Deriving the higher-order Weinberg sum rules from the OPE for correlation functions, one typically encounters the following problem: From the OPE side, the condensate terms have anomalous dimensions starting from dimension 6, while from the resonance side, the anomalous dimensions are absent by construction, thus the question arises about the correctness of equating both sides. In this section, we will argue that the sum rules are closely related to the Lagrangian approach, in fact they can be derived from this approach assuming the asymptotic chiral symmetry restoration at large momentum transfer. The problem with anomalous dimensions can be escaped in this case.

At present the approximation of narrow resonances has a solid theoretical foundation – it is equivalent to the large-N_c (or planar) limit of QCD [82,83], where the quantity $g^2 N_c$ is kept fixed, g is the QCD coupling constant. In the planar limit, the meson states are narrow (the meson decay width behaves as $\Gamma \sim O(1/N_c)$) and weakly interacting. Confining ourselves to this limit, we may therefore regard the mesons as almost free particles. In addition, we may suppose that all states with fixed quantum numbers are generated by a universal external source. The Lagrangian of free vector and axial-vector fields generated by external sources $J^{V,A}$ is,

$$L = \sum_n \sum_{\varphi=V,A} \left\{ \frac{1}{4}\left(\partial_\mu \varphi^a_{n,\nu} - \partial_\nu \varphi^a_{n,\mu} \right)^2 - \frac{1}{2} m^2_{\varphi,n} \left(\varphi^a_{n,\mu} \right)^2 + \varphi^a_{n,\mu} J^{\varphi,a}_\mu \right\}. \tag{1}$$

Here a refers to the isospin index. For the time being we neglect the Chiral Symmetry Breaking (CSB). Assume that each field in (1) corresponds to a conserved current. There exists then identity "current = field" in the sense that they act identically in matrix elements. Define the corresponding currents as

$$j^{\varphi,a}_{n,\mu} = F_{\varphi,n} m_{\varphi,n} m_{\varphi,n} \varphi^a_{n,\mu}, \tag{2}$$

where $F_{\phi,n}$ are the electromagnetic decay constants. The conservation of current (2) can be easily checked by taking derivative from the corresponding equation of motion for Lagrangian (1) using $\partial_\mu J^{\varphi,a}_\mu = 0$ due to the isospin conservation. This is equivalent to the Lorentz gauge condition, $\partial_\mu \varphi^a_{n,\mu} = 0$, which is needed to obtain the standard Klein-Gordon-Fock equation with external source,

$$\left(\Box - m^2_{\varphi,n} \right) \varphi^a_{n,\mu} = -J^{\varphi,a}_\mu. \tag{3}$$

Now we can construct a conserved current to which all states with fixed quantum numbers are coupled,

$$j_{\mu}^{\varphi,a} = \sum_{n} j_{n,\mu}^{\varphi,a}. \tag{4}$$

The fields $\varphi_{n,\mu}^{a}$ correspond to physical states Φ_{n}^{a}, which are created when one acts by current (4) on the physical vacuum. Since the latter is electrically neutral, the procedure has to be realized for the third isospin component, the normalized expression is

$$\langle 0 \mid j_{\mu}^{\varphi,3}(x) \mid \Phi_{n}^{3} \rangle = F_{\varphi,n} m_{\varphi,n} \frac{e^{ipx}}{\sqrt{2p_0}} \in_{\mu} . \tag{5}$$

The decay constant $F_{\phi,\, n}$ may be related to observable quantities. For instance, if we associate V_1 with the ρ-meson, matrix element (5) can be estimated from the decay $\rho^0 \to e^+e^-$ (see Eq. (84)).

In real world the CSB occurs, this phenomenon was proven rigorously in the large-N_c limit of QCD under some assumptions [84]. The main impact of CSB for us is the appearance of pions, as a result the axial current is not conserved anymore. Let us introduce the pions by means of the Partial Conservation of Axial Current (PCAC) hypothesis, i.e. we perform the shift

$$A_{1,\mu}^{a} \to A_{1,\mu}^{a} - f_{\pi} \partial_{\mu} \pi^{a}, \tag{6}$$

in the expressions above. Here f_π is the weak pion decay constant, $f_\pi = 92.4$ MeV, in the chiral limit $f_\pi \approx 87$ MeV. The corresponding axial current is then conserved in the chiral limit, $m_\pi = 0$, which we shall adopt. Furthermore, we will make use of quite common assumption of generalized PCAC, i.e. the covariant derivative of pion field is mixed with the axial field $A_{1,\mu}^{a}$ only, the axial fields $A_{n,\mu}^{a}$, $n > 1$, are supposed to correspond to heavier states ("radial" excitations).

The part of Lagrangian (1) related to the interaction with the external sources can be rewritten (factor constant) as

$$L_{\text{int}} = J_{\mu}^{V,a} j_{\mu}^{V,a} + J_{\mu}^{A,a} j_{\mu}^{A,a}. \tag{7}$$

Since only planar diagrams survive in the large-N_c approximation, to the leading order in the large-N_c counting the amplitudes calculated from (7) will be saturated by the one-particle exchanges, the relevant diagrams are displayed in Fig. 1.

The Fourier transforms of the corresponding vector and axial-vector amplitudes are

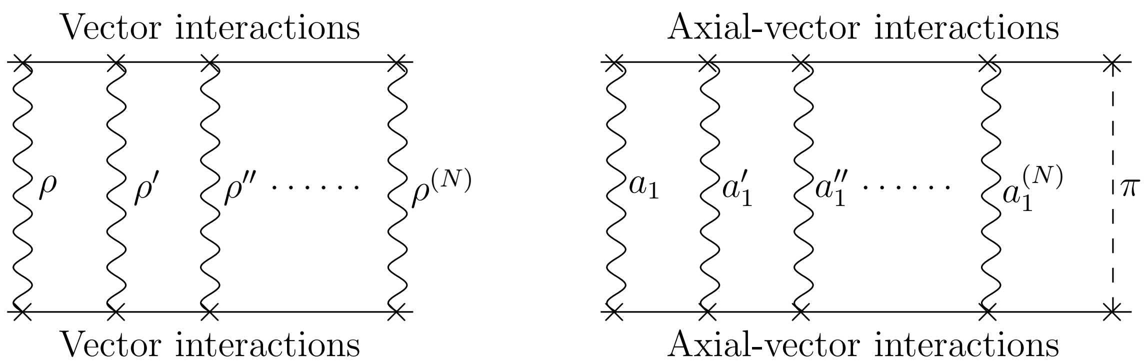

Figure 1
The diagrams of meson exchanges for Lagrangian (7), where the vector and axial-vector states are supposed to be the ρ and a_1 mesons, respectively. The "radial" excitations are marked by primes.

$$W_V(p) = C\, J_\mu^{V,a}(p)\left(\sum_n F_{V,n}^2 m_{V,n}^2 \frac{-g_{\mu\nu}+\frac{p_\mu p_\nu}{m_{V,n}^2}}{p^2-m_{V,n}^2+i\varepsilon}\right)J_\nu^{V,a}(p), \tag{8}$$

$$W_A(p) = C\, J_\mu^{A,a}(p)\left(\sum_n F_{A,n}^2 m_{A,n}^2 \frac{-g_{\mu\nu}+\frac{p_\mu p_\nu}{m_{A,n}^2}}{p^2-m_{A,n}^2+i\varepsilon}+\frac{f_\pi^2 p_\mu p_\nu}{p^2+i\varepsilon}\right)J_\nu^{A,a}(p), \tag{9}$$

where C is a constant. In such a form, there is a seeming discrepancy with the Ward identities, however, this drawback can be always cured, e.g., by an additive renormalization of the amplitudes at $p^2 = 0$, so we shall not bother about it in what follows. Let us separate the transverse and longitudinal parts with the help of identity

$$\frac{-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}}{p^2-m^2+i\varepsilon} = \frac{-g_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}}{p^2-m^2+i\varepsilon}+\frac{p_\mu p_\nu}{m^2 p^2}+O(\varepsilon), \tag{10}$$

which leads to

$$W_V(p) = C\, J_\mu^{V,a}(p) \left\{ \sum_n F_{V,n}^2 m_{V,n}^2 \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}}{p^2 - m_{V,n}^2 + i\varepsilon} + \frac{p_\mu p_\nu}{p^2} \sum_n F_{V,n}^2 \right\} J_\nu^{V,a}(p) + O(\varepsilon), \tag{11}$$

$$W_A(p) = C\, J_\mu^{A,a}(p) \left\{ \sum_n F_{A,n}^2 m_{A,n}^2 \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}}{p^2 - m_{A,n}^2 + i\varepsilon} + \frac{p_\mu p_\nu}{p^2} \left(\sum_n F_{A,n}^2 + f_\pi^2 \right) \right\} J_\nu^{A,a}(p) + O(\varepsilon). \tag{12}$$

The exact chiral symmetry implies that the expressions in the braces of Eqs. (11) and (12) have to be equal at all p. This is possible only if $m_{V,n} = m_{A,n}$, $F_{V,n} = F_{A,n}$, $f_\pi = 0$, which is far from reality. Instead of exact chiral symmetry, one usually imposes an asymptotic chiral symmetry at large energies.

Let us interpolate the difference of vector and axial-vector amplitudes at large four-momentum $|p|$ by the following Taylor expansion,

$$W_V(p) - W_A(p) = \sum_{k=0}^{\infty} \frac{\Delta k}{p^{2k}}. \tag{13}$$

We have taken into account that the Lorentz invariance dictates the dependence on p^2 in the final answer. The quantities Δ_k are unknown constants to be determined. The requirement that the vector and axial-vector interactions are indistinguishable at large p leads to $\Delta_0 = 0$, hence, to the equality of longitudinal parts in Eqs. (11) and (12), this is the first nontrivial constraint from the asymptotic chiral symmetry. Consider difference (13) at $O(p^{-2})$. In the limit of very large p, this difference is identical to the difference of vector and axial-vector amplitudes in a theory where only massless vector and axial-vector mesons are exchanged. If the vector and axial-vector interactions are indistinguishable at large p in the latter theory, the difference above is zero. This analogy suggests that we may assume a more strong asymptotic chiral symmetry and admit $\Delta_1 =$

0. Comparing Eqs. (11) and (12) with Eq. (13), the following set of asymptotic sum rules can be written,

$$\sum_n F_{V,n}^2 - \sum_n F_{A,n}^2 - f_\pi^2 = 0, \tag{14}$$

$$\sum_n F_{V,n}^2 m_{V,n}^2 - \sum_n F_{A,n}^2 m_{A,n}^2 = 0, \tag{15}$$

$$\sum_n F_{V,n}^2 m_{V,n}^{2k} - \sum_n F_{A,n}^2 m_{A,n}^{2k} = \Delta_k, \quad k = 2,3,4,\ldots \tag{16}$$

Sum rules (14) and (15) are the Weinberg sum rules [1] generalized to the case of arbitrary number of states. Some general properties of set of equations (14)-(16) were studied in [85], where these sum rules were derived from the OPE of two-point quark current correlators [26]. It should be emphasized the distinction of the OPE-based ideology from the one adopted here. The quantities Δ_k are usual finite numbers in our approach, while in the OPE they are related to the condensates of appropriate dimension multiplied by the coefficient functions calculated from QCD by means of the perturbation theory (the ALEPH/OPAL data on the V and A spectral functions from τ decays was used for numerical estimations of quantities Δ_k up to dimension 18, see [48] for a review). The relation with the fundamental theory is an advantage of the OPE-based methods, in particular, the restrictions $\Delta_0 = 0$ and $\Delta_1 = 0$ follow automatically in the chiral limit. However, from the point of view of sum rules (16), the OPE has two shortcomings. First, the OPE represents, at best, an asymptotic expansion with zero radius of convergence, therefore the calculation of Δ_k at large k (in practice, at $k \gtrsim 4$) is not reliable because the divergence sets in. Second, as mentioned above the condensate terms have an anomalous dimension starting from dimension 6 while the l.h.s. of Eq. (16) does not have, this circumstance caused a critics of such sum rules recently [86].

Another derivation of generalized Weinberg sum rules, Eqs. (14) and (15), which does not use the correlation functions, was proposed in [73], where the method consisted in analysis of certain Current Algebra commutation relations in infinite momentum frame.

3. Sum rules: correlator approach

Originally the sum rules under consideration were derived in [1] from some asymptotic restrictions on correlation functions with subsequent saturation by narrow states (an earlier application of asymptotic equality for the V and A correlators exists in the literature [87], where the rate for $\omega^0 \to \pi^0 + \gamma$ was calculated from one of sum rules emerging in the infinite energy limit; we are grateful to Prof. S. Gasiorowicz for this remark). We will consider a generalization of this method to the case of arbitrary number of narrow states and discuss the underlying physics.

Introduce the vector and axial-vector two-point correlation functions,

$$\Pi^{V}_{\mu\nu}(p) = i\int d^4x\, e^{ipx} \left\langle 0 \left| T\left(j^{V,a}_{\mu}(x) j^{V,a}_{\nu}(0) \right) \right| 0 \right\rangle, \tag{17}$$

$$\Pi^{A}_{\mu\nu}(p) = i\int d^4x\, e^{ipx} \left\langle 0 \left| T\left(j^{A,a}_{\mu}(x) j^{A,a}_{\nu}(0) \right) \right| 0 \right\rangle. \tag{18}$$

The V and A currents are commonly interpolated by the following quark bilinears,

$$j^{V,a}_{\mu} = \bar{q}\gamma_{\mu}\frac{\tau^a}{2}q, \quad j^{A,a}_{\mu} = \bar{q}\gamma_{\mu}\gamma_5\frac{\tau^a}{2}q, \tag{19}$$

where τ^a are isospin Pauli matrices. For the time being we do not need explicit expression for these currents. Consider the spectral representation for the vector correlator,

$$\Pi^{v}_{\mu\nu}(p) = \int_0^\infty dm^2 \rho^V(m^2) \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2 + i\varepsilon} + \text{S.T.}, \tag{20}$$

where $\rho^V(m^2)$ is spectral density, meaning an amplitude of probability that external vector field V creates an excitation with the mass m. The letters S.T. denote a Schwinger term, which emerges because the spectral representation is covariant, while the chronological T-product in Eq. (17) is not covariant [88,89]. Separating the transverse and longitudinal parts and substituting a manifest expression for the Schwinger term in our case [88], we obtain,

$$\begin{aligned}\Pi^{V}_{\mu\nu}(p) = {} & \int_0^\infty dm^2 \rho^V(m^2) \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}}{p^2 - m^2 + i\varepsilon} + \frac{p_\mu p_\nu}{p^2}\int_0^\infty dm^2 \frac{\rho^V(m^2)}{m^2} + \\ & + \delta_{0\mu}\delta_{0\nu}\int_0^\infty dm^2 \frac{\rho^V(m^2)}{m^2} + O(\varepsilon).\end{aligned} \tag{21}$$

Analogous expression takes place for the axial-vector case,

$$\begin{aligned}\Pi^{A}_{\mu\nu}(p) = {} & \int_0^\infty dm^2 \rho^A(m^2) \frac{-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}}{p^2 - m^2 + i\varepsilon} + \frac{p_\mu p_\nu}{p^2}\left(\int_0^\infty dm^2 \frac{\rho^A(m^2)}{m^2} + f_\pi^2\right) + \\ & + \delta_{0\mu}\delta_{0\nu}\left(\int_0^\infty dm^2 \frac{\rho^A(m^2)}{m^2} + f_\pi^2\right) + O(\varepsilon).\end{aligned} \tag{22}$$

Let us require the asymptotic chiral symmetry [2],

$$\lim_{|p|\to\infty}\left(\Pi^{V}_{\mu\nu}(p)-\Pi^{A}_{\mu\nu}(p)\right)=0. \tag{23}$$

The first Weinberg sum rule then follows from the longitudinal part $p_\mu p_\nu/p^2$,

$$\int_0^\infty dm^2\,\frac{\rho^V(m^2)}{m^2}=\int_0^\infty dm^2\,\frac{\rho^A(m^2)}{m^2}+f_\pi^2. \tag{24}$$

The assumption that chiral symmetry is stronger leads to nullifying the transverse part at $(-g_{\mu\nu} + p_\mu p_\nu/p^2)/(p^2 - m^2)$, this is the second Weinberg sum rule,

$$\int_0^\infty dm^2\rho^V(m^2)=\int_0^\infty dm^2\rho^A(m^2). \tag{25}$$

In general case, the transverse parts satisfy the following condition,

$$\lim_{p\to\infty}p^{2k}\left(\Pi^{V}_{\mu\nu}(p)-\Pi^{A}_{\mu\nu}(p)\right)=\Delta_k,\quad k=1,2,\ldots, \tag{26}$$

where Δ_k are some constants of mass dimension $2k + 2$. The corresponding asymptotic sum rules are

$$\int_0^\infty dm^2m^{2(k-1)}\rho^V(m^2)-\int_0^\infty dm^2m^{2(k-1)}\rho^A(m^2)=\Delta_k, \tag{27}$$

which reduce to the second Weinberg sum rule (25) for $k = 1$, $\Delta_1 = 0$. Using the definition of spectral density,

$$\left(-g_{\mu,\nu}p^2+p_\mu p_\nu\right)\rho(p^2)=(2\pi^3)\sum_n\delta^{(4)}(p-p_n)\langle 0\,|\,j^3_\mu(0)\,|\,n\rangle\langle n\,|\,j^3_\nu(0)\,|\,0\rangle, \tag{28}$$

one can saturate the obtained equations by narrow resonances,

$$\rho^{V,A}(m^2)=\sum_n F^2_{V,A;n}\delta\left(m^2-m^2_{V.A;n}\right), \tag{29}$$

and get the sum rules considered above, Eqs. (14)–(16).

An interesting consequence of the correlator formalism is that the first Weinberg sum rule follows also from the requirement of cancellation of Schwinger terms in the $\Pi_V - \Pi_A$ difference [88], this fact is evident from Eqs. (21) and (22). The cancellation of Schwinger terms is quite natural as long as the $\Pi_V - \Pi_A$ difference is related to observable quantities. Thus, one arrives at a kind of equivalence between the asymptotic chiral symmetry and the cancellation of Schwinger terms. A question emerges, which principle is more fundamental?

The Schwinger terms emerge in equal-time current commutators when one defines a current as the limit

$$j_\Gamma(x_0,\mathbf{x}) \equiv \lim_{\varepsilon\to 0} \bar{q}(x_0,\mathbf{x}+\varepsilon)\Gamma q(x_0,\mathbf{x}-\varepsilon), \tag{30}$$

where j_Γ is the current corresponding to the gamma-matrix structure Γ. These terms can be both operators and c-numbers, generally speaking, their form is model-dependent. The appearance of Schwinger terms is an inescapable consequence of Lorentz invariance and of positive definiteness for probability, otherwise the theory is trivial [88,89]. This observation suggests that the cancellation of Schwinger terms is likely more fundamental requirement than the cancellation of the longitudinal parts at large momentum. In addition, the former cancellation should take place at all momenta as it is momentum-independent. The given property explains why the first Weinberg sum rule is much better fulfilled in the phenomenology than the second one, the reason seems to be that it is not only asymptotic sum rule – its validity extends beyond the high-energy domain. Indeed, provided the equality of Schwinger terms in spectral representation (20) and in its axial-vector analogue, the first Weinberg sum rule equally emerges if one takes the limit $|p| \to 0$ in condition (23). On the other hand, it should be mentioned that the problem with the Schwinger terms can be escaped from the very beginning if we subtract automatically the contribution due to contact terms in definitions (17) and (18) performing thereby an additive renormalization. Such an operation gives the so-called T^*-product which is gauge-invariant and Lorentz covariant. This could be a better starting point, but our aim was to demonstrate some delicate points with the Weinberg sum rules.

Asymptotic sum rules at large momentum can be supplemented by low-energy sum rules at vanishing momentum, which are nothing but an extension of sum rules (27) to negative k provided by the requirement

$$\lim_{|p|\to 0} p^{2k}\left(\Pi^V_{\mu\nu}(p) - \Pi^A_{\mu\nu}(p)\right)\Big|_{\text{f.p.}} = \Delta_k, \quad k = -1,-2,\ldots, \tag{31}$$

where "f.p." means that only the final part is retained. The constant Δ_{-1} is known from the phenomenology [90],

$$\Delta_{-1} = -4\bar{L}_{10}, \tag{32}$$

where $\bar{L}_{10}$ is the scale independent part of the coupling of the relevant operator in the $O(p^4)$ effective chiral Lagrangian of QCD [91]. This constant can be expressed by means of the following combination of hadronic parameters,

$$\bar{L}_{10} = -\frac{1}{4}\left(\frac{1}{3} f_\pi^2 \langle r_\pi^2 \rangle - \mathcal{F}_A \right), \tag{33}$$

where $\langle r_\pi^2 \rangle$ is the electromagnetic mean mass squared radius of the charged pions and $\mathcal{F}_A$ is the axial-vector coupling. Relation (33) is nothing but the Das-Mathur-Okubo low-energy theorem [10].

A general property of all considered sum rules is that they depend on the four-momentum cutoff μ through the number of included resonances only. Due to the equality $\Delta_1 = 0$ in the second Weinberg sum rule, another sum rule with the given property can be written,

$$\int_0^\infty dm^2 m^2 \ln\left(\frac{m^2}{\mu^2} \right) \rho^V(m^2) - \int_0^\infty dm^2 m^2 \ln\left(\frac{m^2}{\mu^2} \right) \rho^A(m^2) = \bar{\Delta}_1. \tag{34}$$

The constant $\bar{\Delta}_1$ turns out to be related with a remarkable physical observable, the electromagnetic pion mass difference [9], $\bar{\Delta}_1 \sim m_{\pi^\pm}^2 - m_{\pi^0}^2$, we will exploit this relation later.

4. Weinberg-like sum rule equations: solutions for particular cases

We are going to solve the system of equations (14)–(16) for some typical cases. Partly, our analysis may be regarded as a revision and extension of results obtained in [85]. We, however, place quite different emphasis – we are interested in maximally degenerate solutions (aside from the trivial ones) for excited V and A mesons and in overall correspondence of different possibilities to the actual experimental data.

4.1 One Vector and One Axial-vector

In this subsection we revisit the standard Weinberg ansatz and clarify its modern status.

It is convenient to work with the dimensionless quantities,

$$X_{V,A} = \frac{F_{V,A}^2}{f_\pi^2}, \tag{35}$$

$$Y_{V,A} = \frac{M_{V,A}^2}{M_\rho^2}. \tag{36}$$

Experimentally [92] (in MeV),

$$F_V = 154 \pm 8,\ F_A = 123 \pm 25,\ M_V = 776,\ M_A = 1230 \pm 40, \tag{37}$$

and $f_\pi \approx 87$ MeV in the chiral limit which will be used in the sequel for our estimations, we remind that in real world $f_\pi = 92.4$ MeV. We will associate the V state with the ρ-meson, the first three sum rules look then as follows,

$$\begin{cases} X_\rho - X_A = 1 \\ X_\rho - X_A Y_A = 0 \\ X_\rho - X_A Y_A^2 = \Delta, \end{cases} \tag{38}$$

where the notation $\Delta = \Delta_2$ is adopted (see Eq. (16)). The first two sum rules were first proposed by Weinberg [1]. They give for the third sum rule,

$$\Delta = X_\rho (1 - Y_A) = -\frac{X_\rho}{X_\rho - 1}. \tag{39}$$

From OPE [26],

$$\Delta_{\mathrm{OPE}} = -\frac{4\pi\alpha_s \langle \bar{q}q \rangle^2}{M_\rho^4 f_\pi^2}. \tag{40}$$

At 1 GeV one has for the strong coupling and the quark condensate,

$$\alpha_s \approx 0.5, \quad \langle \bar{q}q \rangle \approx (-235\ \mathrm{MeV})^3. \tag{41}$$

The phenomenological and experimental values above yield the following estimate,

$$\Delta_{OPE} \approx -0.3. \tag{42}$$

Another two sum rules follow from the electromagnetic pion mass difference [9],

$$m_{\pi^\pm} - m_{\pi^0} = -\frac{C_0}{f_\pi^2} \int_0^\infty dm^2 m^2 \ln\left(\frac{m^2}{\mu^2} \right) \left(\rho^V(m^2) - \rho^A(m^2) \right), \tag{43}$$

Where

$$C_0 = \frac{3\alpha}{8\pi m_\pi}, \tag{44}$$

(α is the fine structure constant and μ denotes an arbitrary scale) and the scale independent constant $\bar{L}_{10}$ of effective Chiral Lagrangian [91],

$$\bar{L}_{10} = -\frac{1}{4}\int_0^\infty dm^2 \frac{\rho^V(m^2) - \rho^A(m^2)}{m^2}. \tag{45}$$

The experiment [92] and the chiral phenomenology [93] yield for $m_{\pi^\pm} - m_{\pi^0}$ and $\bar{L}_{10}$ respectively

$$m_{\pi^\pm} - m_{\pi^0} = 4.6 \text{ MeV}, \quad \bar{L}_{10} = -(5.5 \pm 0.7)\cdot 10^{-3}. \tag{46}$$

Saturating the spectral densities by one resonance plus continuum in Eqs. (43) and (45), we obtain

$$\begin{aligned} m_{\pi^\pm} - m_{\pi^0} &= -\frac{C_0}{f_\pi^2}\left(M_\rho^2 F_\rho^2 \ln\frac{M_\rho^2}{\mu^2} - M_A^2 F_A^2 \ln\frac{M_A^2}{\mu^2} \right) \\ &= C_0 M_\rho^2 X_\rho \ln\frac{X_\rho}{X_\rho - 1}, \end{aligned} \tag{47}$$

$$\bar{L}_{10} = -\frac{f_\pi^2}{4M_\rho^2}\left(\frac{X_\rho}{Y_\rho} - \frac{X_A}{Y_A} \right) = -\frac{f_\pi^2}{4M_\rho^2}\left(2 - \frac{1}{X_\rho} \right), \tag{48}$$

where the first and the second Weinberg sum rules have been used.

To predict concrete numerical values one should fix an input parameter, say X_ρ. Originally [1] Weinberg assumed the KSFR relation [94,95],

$$X_\rho = 2, \tag{49}$$

resulting in

$$X_A = 1, \ Y_A = 2. \tag{50}$$

We will refer to this ansatz as the "Weinberg" one. It predicts (in the chiral limit),

$$\Delta = -2, \quad \bar{L}_{10} \approx -4.7 \cdot 10^{-3}, \tag{51}$$

and (in MeV),

$$F_\rho \approx 123, \quad F_A \approx 87, \quad M_A \approx 1100, \quad m_{\pi^\pm} - m_{\pi^0} \approx 5.2. \tag{52}$$

Identifying the A state with the a_1-meson, the Weinberg ansatz was widely used in the literature for matching conditions and other purposes.

We would like to note, however, that the status of the KSFR relation caused much discussions in the literature. In particular, its original derivation was revised in [96-101], the main lesson was that it required more *ad hoc* assumptions in comparison with the ones made in [94,95]. Notably, the modern experimental data (37) favors rather to the ansatz

$$X_\rho = 3, \tag{53}$$

leading to

$$X_A = 2, \quad Y_A = \frac{3}{2}. \tag{54}$$

We will refer to this ansatz as the "experimental" one. It predicts (in the chiral limit),

$$\Delta = -\frac{3}{2}, \quad L_{10} \approx -5.2 \cdot 10^{-3}, \tag{55}$$

and (in MeV),

$$F_\rho \approx 151, \quad F_A \approx 123, \quad M_A \approx 950, \quad m_{\pi^\pm} - m_{\pi^0} \approx 4.6. \tag{56}$$

Comparing the predictions with the corresponding experimental and phenomenological values given above, one can see immediately that the experimental ansatz works better substantially than the Weinberg one: Lowering M_A by 15% (which is within the large-N_c accuracy), one amends noticeably all other five quantities.

Finally, it should be mentioned that the degenerate case, which would mean here $Y_A = 1$, cannot be obtained within the considered ansatz.

4.2 One Vector and No Axial-vector

As was discussed in Introduction, it makes sense to consider the ansatz without the axial-vector meson, the "would-be" chiral partner of the vector state is then the pion (the given possibility was not analyzed in [85]). This case amounts to setting $X_A = 0$ in the formulae of the previous subsection. First of all, it is obvious that only the first sum rule in Eqs. (38) can be satisfied with this ansatz, yielding $X_\rho = 1$, i.e. $F_\rho = f_\pi$. The physical sense of this result is that had the "vector manifestation" been exactly realized, only the first sum rule would have survived because the ρ-meson is then massless. The value of constant $\bar{L}_{10}$ from Eq. (48) is

$$\bar{L}_{10} = -\frac{f_\pi^2}{2M_\rho^2} \approx -6.3 \cdot 10^{-3}. \tag{57}$$

which represents a quite reasonable estimate. Taking into account $X_\rho = 1$, now the electromagnetic mass difference from Eq. (47),

$$m_{\pi^\pm} - m_{\pi^0} = C_0 M_\rho^2 \ln \frac{\mu^2}{M_\rho^2}, \tag{58}$$

depends explicitly on the cutoff μ. Treating the cutoff as an input parameter, one can achieve any value for $m_{\pi^\pm} - m_{\pi^0}$, in particular, the experimental one, 4.6 MeV, is implemented at $\bar{\mu} \approx 1435$ MeV. In this respect, it is not surprising that $m_{\pi^\pm} - m_{\pi^0}$ can be calculated without use of the a_1-meson, see e.g. [102]. On the other hand, the cutoff should not exceed the mass of the higher resonance, otherwise the latter has to be included explicitly. Since the higher resonances give a sizeable contribution to $m_{\pi^\pm} - m_{\pi^0}$ [103], the physical interpretation for $\bar{\mu}$ seems to be the following: It indicates indirectly on the typical scale of the higher meson excitations such as $a_1(1230)$ and $\rho(1450)$. The result of the previous subsection is reproduced if we identify the cutoff with the mass of the lowest A-meson.

In summary, the ρ- π ansatz is not senseless within the sum rules, it seems to be indeed the simplest ansatz consistent, to a certain extent, with the phenomenology.

4.3 Two Vectors and One Axial-vector

The ansatz with two V-mesons and one A-state was considered in [85] and also in [47,67]. We will reexamine this possibility following the emphasis expressed in the beginning of this section.

Extending sum rules (38) by the second vector state, we arrive at

$$\begin{cases} X_\rho - X_A + X_V = 1 \\ X_\rho - X_A Y_A + X_V Y_V = 0 \\ X_\rho - X_A Y_A^2 + X_V Y_V^2 = \Delta. \end{cases} \tag{59}$$

It is natural to associate the second vector state with the ρ (1450)-meson, whose mass is [92]

$$M_V = 1459 \pm 11 \text{ MeV}, \tag{60}$$

while its decay constant F_V is unknown. The solution of system (59) can be written as

$$\begin{cases} X_A = \dfrac{((X_\rho - 1)Y_V - X_\rho)^2}{(X_\rho - 1)Y_V^2 - 2X_\rho Y_V + X_\rho - \Delta} \\ X_V = \dfrac{X_\rho + (X_\rho - 1)\Delta}{(X_\rho - 1)Y_V^2 - 2X_\rho Y_V + X_\rho - \Delta} \\ Y_A = \dfrac{X_\rho (Y_V - 1) + \Delta}{(X_\rho - 1)Y_V - X_\rho}. \end{cases} \tag{61}$$

These solutions are positive at

$$\Delta \geq -\frac{X_\rho}{X_\rho - 1}, \tag{62}$$

with the inequality,

$$Y_V \geq Y_A, \tag{63}$$

being always maintained. Solutions are not valid when the corresponding denominator is zero, the corresponding special point is (we are interested in the physical case $Y_V > 1$ only),

$$(Y_V)_s = (Y_A)_s = \frac{X_\rho + \sqrt{X_\rho + (X_\rho - 1)\Delta}}{X_\rho - 1}. \tag{64}$$

Physically we expect that the ρ-meson dominates over heavier resonances, this leads to the inequalities

$$X_A \leq X_\rho, \; X_V \leq X_\rho. \tag{65}$$

The second inequality holds automatically because $X_V \leq 1$ due to the first sum rule in system (59), while physically $X_\rho \geq 2$. The first inequality results in the lower bound on Y_V,

$$Y_V \geq \frac{X_\rho}{X_\rho - 1}\left(1 + \sqrt{1 + \frac{X_\rho - 1}{X_\rho}\Delta}\right). \tag{66}$$

To estimate the ensuing restriction on the mass M_V we can admit $\Delta = 0$. Then one has $M_V \geq$ 1550 MeV for $X_\rho = 2$ and $M_V \geq 1340$ MeV for $X_\rho = 3$. Thus, the physical value of $\rho(1450)$-meson (60) mass is incompatible with the KSFR relation within the ansatz under consideration.

The possibility $Y_V = Y_A$ is realized at finite X_V and X_A only if

$$\Delta = -\frac{X_\rho}{X_\rho - 1}, \tag{67}$$

while in this case

$$Y_V = Y_A = \frac{X_\rho}{X_\rho - 1}, \tag{68}$$

i.e. Eqs. (67) and (68) yield

$$\Delta + Y_A = 0. \tag{69}$$

The impossibility to provide $\Delta = 0$ for degenerate case in realistic situations recurs at introducing more resonances. For instance, if we add a pair of resonances (ρ" and a_1' mesons) with equal masses then one can show that the condition $\Delta = 0$ may be adjusted only when $X_A > 2(X_\rho + 1)$ while we expect $X_A \leq X_\rho$.

Finally we note that the considered possibility for the degenerate case, $Y_V = Y_A$, was missed in the analysis [85]. The solution of system (61) was written in [85] as (in our notations)

$$\begin{cases} X_\rho = \dfrac{Y_A Y_V + \Delta}{(Y_A - 1)(Y_V - 1)} \\ X_A = -\dfrac{Y_V + \Delta}{(Y_A - 1)(Y_V - Y_A)} \\ X_V = \dfrac{Y_A + \Delta}{(Y_V - 1)(Y_V - Y_A)}. \end{cases} \tag{70}$$

The conclusion made in [85] was that when relation (69) takes place then $X_V = 0$ and the highest vector state decouples. Evidently, this is not true if $Y_V = Y_A$, i.e. when the V and A mesons are exactly degenerate.

4.4 Arbitrary Finite Number of States

The case of arbitrary, but finite, number of states is very model dependent. The general analysis of this case performed in [85] was essentially based on the following simplification: The number of unknown variables $X_{V,A}$ is equal to the number of sum rules under consideration, the system of equations for $X_{V,A}$ is then linear. Clearly, this is a model assumption, the real-life physics should not depend on the way one solves the sum rule equations. We are interested in the most degenerate case other than the complete degeneracy of radial excitations in masses and residues, i.e. when all is determined by the ground states only. The physical spectrum looks like a perturbed linear (in masses square) spectrum. Staying within the linear parametrization, it is possible to satisfy the two Weinberg sum rules with arbitrary number of states by fine-tuning of residues, but hardly possible to satisfy the higher-order sum rules, this would require the introduction of model dependent corrections to the linear behaviour. As an example close to the real-life physics we give the following ansatz, which satisfies the two Weinberg sum rules identically,

$$\Pi^V(p^2) = \frac{2F_\rho^2}{p^2 - m_\rho^2 + i\varepsilon} + 2\sum_{n=1}^{N} \frac{F_{V,n}^2}{p^2 - m_{V,n}^2 + i\varepsilon} + \text{P.C.}, \tag{71}$$

$$\Pi^A(p^2) = \frac{2f_\pi^2}{p^2 + i\varepsilon} + 2\sum_{n=1}^{N} \frac{F_{A,n}^2}{p^2 - m_{A,n}^2 + i\varepsilon} + \text{P.C.}, \tag{72}$$

where P.C. means "perturbative continuum", the $\Pi^{V,A}(p^2)$ are defined as

$$\Pi_{\mu\nu}^{V,A}(p) = \left(-g_{\mu\nu}p^2 + p_\mu p_\nu\right)\Pi^{V,A}(p)^2, \tag{73}$$

and the masses and residues are as follows

$$F_\rho^2 = 2f_\pi^2, \quad F_{V,n}^2 = \begin{cases} 2f_\pi^2, & n < N, \\ f_\pi^2, & n = N \end{cases}, \quad m_{V,n}^2 = m_\rho^2(2+2n); \tag{74}$$

$$F_{A,n}^2 = 2f_\pi^2, \quad m_{A,n}^2 = m_\rho^2(1+2n). \tag{75}$$

In this example the ρ-meson is singled out, its residue is in accord with the KSFR relation, the universal slope $2m_\rho^2$ agrees with the phenomenology and some models (see [104] for discussions) as well as the universal residue $2f_\pi^2$. We have made here a minimal manipulation with residues – the residue of the highest vector state is two times less than the universal one. The physical interpretation could be given the following: The resonance of mass $m_{V,N}^2 = m_\rho^2(2+2N)$ is the heaviest in the system, if one cuts off at $\mu_{\text{cut}} = m_{V,N}$ then the half of its decay width (namely, the right half from the position of resonance) is thrown away, in the narrow-width approximation this loss of information can be mimicked by halfing the residue.

We could not achieve the completely degenerate case, $m_{V,N}^2 = m_{A,N}^2$ at least in asymptotics, a removal of degeneracy seems to be unavoidable if one likes to get rid of nonlinear corrections. It is interesting to note that the mass spectrum of the ansatz above resembles that of old dual models [105] (a somewhat similar model, but for infinite number of states, was considered in [106]).

In the case of arbitrary number of states there is one subtlety which is practically always ignored in the sum rules under consideration – there exist two kinds of V-mesons, the S-wave and D-wave ones, and both couple to the interpolating V-current (19) [72]. The only exception is paper [74], where the problem was addressed for the case of infinite number of resonances, but the discussions of that paper remain relevant for our case. The doubling of V-states forces to replace the sum in Eq. (71) by the following one,

$$\sum_{n=1}^{N}\frac{F_{V,n}^2}{p^2-m_{V,n}^2+i\varepsilon}\to\sum_{n=1}^{N}\frac{F_{V_S,n}^2}{p^2-m_{V_S,n}^2+i\varepsilon}+\sum_{n=k}^{N}\frac{F_{V_D,n}^2}{p^2-m_{V_D,n}^2+i\varepsilon}, \tag{76}$$

where k is some integer (in practice, $k = 2$ if the ρ-meson is singled out, see Eq. (77)). As to the D-wave vector states, two alternative possibilities were proposed in [74]: (i) the D-wave mesons approach the S-wave V-trajectory, implying asymptotic degeneration; (ii) D-wave V-mesons decouple. The conclusion made in [74] was that the possibility (ii) is more plausible. The recent phenomenological studies, however, invite to reconsider this conclusion. Namely, the spectrum of light nonstrange mesons as a function of radial n and angular momentum L quantum numbers seems to obey a simple relation [71,72,80,107-110],

$$m_{n,L}^2 \sim n + L, \tag{77}$$

the case $L = 0$ corresponds to the S-wave states, while for $L = 2$ one has the D-wave mesons. It is obvious from this relation that an approximate degeneracy of S- and D-wave states occurs, the effect was found by the Crystal Barrel Collaboration [111], although the experimental results are still preliminary. Identifying now

$$m_{V,n}^2 = m_{V_S,n}^2, \quad F_{V,n}^2 = \begin{cases} F_{V_S,n}^2, & n < k, \\ F_{V_S,n}^2 + F_{V_D,n}^2, & n \geq k \end{cases}, \tag{78}$$

we arrive at the standard pattern of resonance saturation, so the usual formulae remain formally valid as if the D-wave states were decoupled. A qualitative argument in favour of D-wave decoupling presented in [74] relied, in essence, on the fact that the vector interpolating current (19) couples to the e^+e^- annihilation, which is a point-like process, hence, the extended objects like the D-wave states should decouple, i.e. their residues vanish rapidly. However, quasiclassical arguments (see, e.g., discussions in [72]) tell us that the orbitally and radially excited mesons are equally extended objects, at least in the hadron string picture, in which the size of meson is defined by its mass only. Thus, once we adopt the large-N_c limit and introduce thereby the high radial excitations regarding them as coupled to a certain local current, we inevitably should encounter the orbital excitations coupled to the same current, under the vector mesons we should then understand the mixture defined above.

5. Infinite number of states

The sum rules dealing with infinite number of V and A states are largely covered in the literature, see e.g. [73,74,104,112-115] and references therein. In this section we give some relevant comments.

First of all, some technical details should be reminded. At large Euclidean momentum Q the asymptotics of the correlation functions is [26] (see definition (73)),

$$\Pi^{V,A}(Q^2) = \frac{N_c}{12\pi^2}\ln\frac{\mu^2}{Q^2} + O\left(\frac{\Lambda_{\text{QCD}}^4}{Q^4}\right), \tag{79}$$

where we neglect $O(\alpha_s)$ correction to the partonic logarithm (the impact

of this correction was studied in [116-120]) and the chiral limit is assumed. In order to obtain additional constraints on hadron parameters, the conventional tactics consists in saturating by resonances the individual correlators $\Pi^{V,A}(Q^2)$ instead of (or along with) the difference $\Pi^V(Q^2)$ - $\Pi^A(Q^2)$. Evidently, to reproduce the logarithm in Eq. (79) the infinite number of resonances is required in the large-N_c limit. Thus, for instance, the vector correlator can be written as (we omit the complication with the D-wave V-states for the reasons explained in Sect. 4.4)

$$\Pi^V(Q^2) = \sum_{n=0}^{\infty}\frac{2F_{V,n}^2}{Q^2 + m_{V,n}^2} + \text{S.C.}, \tag{80}$$

here S.C. means "subtraction constant". It should be noted that the perturbative continuum is not present any more in Eq. (80), this circumstance expresses the quark-hadron duality.

The first approximation to the sum in Eq. (80) is integral over n, this can be explicitly seen via the Euler-Maclaurin summation formula [74,112],

$$\begin{aligned}\sum_{n=0}^{N} f(n) = \int_0^N f(x)dx + \frac{1}{2}[f(0) + f(N)] + \\ + \sum_{k=0}^{\infty}(-1)^k \frac{B_{k+1}}{(2k+2)!}\left[f^{(2k+1)}(N) - f^{(2k+1)}(0)\right],\end{aligned} \tag{81}$$

where $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}, \ldots$ are Bernoulli numbers. Consequently, the par-tonic logarithm can be reproduced if masses and residues are related by

$$F_{V,n}^2 \sim \frac{dm_{V,n}^2}{dn}, \tag{82}$$

which should hold at least at $n \to \infty$. To advance further one needs some ansatz for $m_{V,n}^2$ or $F_{V,n}^2$. There is no experimental information on the residues of highly excited states, but fortunately experiment [111] indicates on the Regge behaviour of masses, $m_n^2 \sim n$. From the theoretical side, the same behaviour is suggested by quantization of quasiclassical meson string (see, e.g., [71,72,121]). These arguments justify the standard use of linear *ansätze* (up to corrections vanishing at $n \to \infty$) in the sum rules under consideration. Relation (82) gives then constant residues. To the best of our knowledge, this is the only reason why one believes that residues are independent of n, at least at large n. The question is whether it is possible to justify qualitatively the constant behaviour of residues independently?

We suggest such argument based on the semiclassical string picture for mesons. Consider a typical pulsating meson string (see, e.g., [72] for details), in the large-N_c limit it never breaks. The mesons decay then due to the electromagnetic interactions of quark and antiquark – if they "meet" each other inside the string and annihilate into the electron-positron pair. The expectation time τ for such an event in the pulsating string is proportional to the string length l, the latter is proportional to the meson mass m. Taking into account that τ is then typical life-time of the meson, one has

$$\Gamma_{V \to e^+e^-} \sim \frac{1}{\tau} \sim \frac{1}{l} \sim \frac{1}{m}. \tag{83}$$

On the other hand, the residues of the vector mesons are related to their electromagnetic widths as

$$\Lambda_{V_n \to e^+e^-} = \frac{4\pi\alpha^2 F_{V,n}^2}{3m_{V,n}} \sim \frac{F_{V,n}^2}{m_{V,n}}. \tag{84}$$

Comparing relations (83) and (84) we conclude that the residues $F_{V,n}^2$ should not depend on n, i.e. they are constant.

Finally, we arrive at a quite unexpected result: Assuming a well-motivated string picture for excited mesons, one is able to argue for the constant residues, then relation (82), which is a consequence of quark-hadron duality, yields the linear mass spectrum, $m_n^2 \sim n$, i.e. this linearity may be deduced from the sum rules without quantization of hadron string.

Another comment concerns derivation of the KSFR relation from the sum rules. This relation is believed to hold in the phenomenology, in the sum rules, however, it is usually imposed from outside. As was argued above, it is natural to expect that V and A meson residues – the electromagnetic decay constants – are a universal constant, $F_{V,A;n}^2 = F^2$. Assuming that this feature extends up to the ρ-meson, the KSFR relation reads $F^2 = 2f_\pi^2$. In principle, we may choose the slope of the linear mass spectrum such that the KSFR relation is reproduced due to Eq. (82), this trick is consistent with the phenomenology [104]. An interesting question rising here is whether it is possible for the KSFR relation to infer from the sum rules in a way weakly dependent on a concrete ansatz for the mass spectrum? We will show that under assumptions above this indeed can be done, the only assumption about the mass spectrum we need is that going up in energy one encounters the resonances in the order V -A -V -A -... . It is a rather weak assumption and consistent with the phenomenology.

Consider the first Weinberg sum rule (14) with arbitrary number N of states and take the limit $N \to \infty$. We should postulate the pattern of pairing for the V and A states. The most frequent assumptions which one commonly uses are: (i) the chiral symmetry of QCD provides equal number of V and A mesons (with the reservation on D-wave vectors above), so the n-th A-meson is paired with the n-th V-meson and the ordering V - A - V - A - ... is thus satisfied; (ii) the ρ-meson is singled out because of the chiral symmetry breaking at low energies, so the n-th A-meson is paired with the $(n + 1)$-th V-meson.

Consider possibility (i). The sum rule (14) is then

$$f_\pi^2 = \lim_{N\to\infty} \sum_{n=1}^{N} \left(F_{V,n}^2 - F_{A,n}^2 \right) = F^2 \sum_{n=1}^{\infty} (-1)^{n+1}. \qquad (85)$$

Here we encounter a usual problem – the limit $N \to \infty$ leads to ill-defined sums and a generalized method of summation is needed in order that such sums make a definite sense. Fortunately, the sum in Eq. (85) is well known in mathematical analysis, there are many ways of generalized summation and practically all of them yield the same result $\frac{1}{2}$ for the sum in question.

Perhaps, the simplest way for defining this sum is

$$\sum_{n=1}^{\infty}(-1)^{n+1} = \lim_{x\to 1-0}\frac{1}{1+x} = \frac{1}{2}. \tag{86}$$

Combining relations (85) and (86) we get the KSFR relation.

Consider possibility (ii). The sum rule (14) takes the form

$$\begin{aligned} f_\pi^2 &= \lim_{N\to\infty}\left(\sum_{n=1}^{N+1}F_{V,n}^2 - \sum_{n=1}^{N}F_{A,n}^2\right) = F_{V,1}^2 - \lim_{N\to\infty}\sum_{n=1}^{N}\left(F_{A,n}^2 - F_{V,n+1}^2\right) \\ &= F^2\left(1 - \sum_{n=1}^{\infty}(-1)^{n+1}\right), \end{aligned} \tag{87}$$

where we have used the assumption of V -A-V -A-... ordering of states in masses. Substituting relation (86) to sum rule (87) we again arrive at the KSFR relation, this universality of result is quite remarkable.

The operations with the divergent sums call for another one comment as they are often present in the sum rules in the large-N_c limit, sometimes this circumstance causes criticism. Naively, such operations should indeed lead to very ambiguous results as long as these sums are ill-defined. This is correct if one throws away any physical content and considers the matter mathematically only. The situation then looks as if we worked with differential equations without boundary conditions, this would be useless for physics. The role of additional assumptions in the sum rules is somewhat similar to that of physical boundary conditions. When imposed correctly, the additional assumptions should always remove ambiguities.

In this regard, the sum $\Sigma(-1)^{n+1}$ is a simple educative example. The answer for this sum depends on a way of grouping the terms, for this reason the problem is not well defined mathematically, but in physics the pattern of grouping is fixed by additional assumption(s). For instance, let us assume that the highly excited V and A states become exactly degenerate, $m_{V,n} = m_{A,n}$, $F_{V,n} = F_{A,n}$ at $n \geq N$. This means that going up in energy, since the scale $m_{V,N}$ the V and A resonances will be coming in pairs. This feature provides the *physical pattern* of grouping the individual contributions in that part of the spectrum,

$$1 - 1 + 1 - 1 + \cdots = (1 - 1) + (1 - 1) + \cdots = 0 + 0 + \cdots = 0, \tag{88}$$

the same pattern will hold in the higher-order sum rules. Such a local conspiracy results in cancellations of the same type as cancellation of V and A perturbative continuums in the difference $\Pi^V(Q^2)$- $\Pi^A(Q^2)$, in this sense one may think of a certain duality between that part of spectrum and perturbative continuum. It should be noted incidentally that assuming some pattern of pairing of states, any regularization of divergent sums has to respect this pattern, otherwise result will be senseless.

Let us assume now a more realistic case – the V and A mesons are not exactly degenerate at any finite N. This means that going up in energy we will be encountering the resonances consecutively state by state, hence, the summation has to be performed in the same way. For validity of this statement it is not necessary to require the V - A - V - A - ... recurrence, if the number of V and A contributions is almost equal (say, if the V and A mesons form approximately degenerate parity doublets), the permutation of some V and A contributions does not change the result. The partial sums of $\Sigma(-1)^{n+1}$ will be then either 0 or 1. The generalized summations yield the averaged value $\frac{1}{2}$. This result is in one-to-one correspondence with the fact that the Euclidean behaviour of correlators is sensitive only to averaged features of the spectral densities, for this reason the generalized summations are usually quite effective tools in the Euclidean domain.

6. Conclusion

We have considered various aspects of generalized Weinberg sum rules. It was argued that when one imposes the asymptotic chiral symmetry on phenomenological Lagrangians, the Weinberg like sum rules follow naturally. The difficulties related to the operator product expansion for correlation functions are problems of a specific derivation of the sum rules rather than problems of the sum rules themselves. In principle, one may use such sum rules including up to infinite number of narrow resonances having forgotten about those problems. In conclusion, we believe that the potential of considered QCD sum rules is not exhausted, further investigations in this direction may lead to interesting results and applications. Hopefully, the present revision of these known sum rules will be useful in this respect.

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