

Research article

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## Ricci Flow Gravity

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Published: 2 October 2007

Received: 29 June 2007

PMC Physics A 2007, 1:3 doi:10.1186/1754-0410-1-3

Accepted: 2 October 2007

This article is available from: <http://www.physmathcentral.com/1754-0410/1/3>

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### Abstract

A theory of gravitation is proposed, modeled after the notion of a *Ricci flow*. In addition to the metric an *independent volume* enters as a fundamental geometric structure. Einstein gravity is included as a limiting case. Despite being a scalar-tensor theory the coupling to matter is different from Jordan-Brans-Dicke gravity. In particular there is no adjustable coupling constant. For the solar system the effects of Ricci flow gravity cannot be distinguished from Einstein gravity and therefore it passes all classical tests. However for cosmology significant deviations from standard Einstein cosmology will appear.

**PACS Codes:** 04.50.+h, 04.20.Cv, 02.40.Hw

### 1 Introduction

A generalization of Einstein's theory of gravity is developed. It has a purely geometric foundation, including in addition to a metric an *independent volume*. Although related to scalar-tensor theories à la Jordan-Brans-Dicke and to string cosmology, it is nevertheless different: the basic field equations are in the form of *Ricci flow equations*, generalized to include matter. Einstein's theory is included as the limiting case of no flow. The volume scalar has two interpretations: geometrically, it is responsible for volume-preservation and physically it obeys a mass-zero real scalar wave equation. This is also the main difference with Jordan-Brans-Dicke theories, where the scalar couples to the trace of the energy-momentum tensor. As a consequence, in general the energy-momentum tensor is not anymore "conservative" in the ordinary sense of  $\nabla \cdot T = 0$ , and point particles do not move anymore on geodesics, having a Newton-Nordström potential. But from Noether's fundamental relation conservation still follows from symmetry. Nevertheless, in "ordinary" solar-system and astrophysical settings, the corrections are negligible: the theory cannot be distinguished from Einstein's and therefore passes all the standard tests. However in a cosmological setting, deviations from standard Einstein gravity are to be expected. This will be the subject of a forthcoming paper.

We will proceed as follows: after this brief introduction, in section 2 the motivations for this kind of extension are discussed. In section 3 a short introduction to volumetric manifolds is given, emphasizing the notion of volume-preservation in section 4. Before establishing the definitive field equations of Ricci flow gravity in section 6, several other choices are discussed in section 5, with emphasis on the main differences with respect to Einstein's and in particular to Jordan-Brans-Dicke theories. Section 7 refers to the physical interpretation of the volume scalar in Ricci flow gravity. Finally, in section 8 the viability of the theory with respect to the standard tests is discussed. The conclusion in section 9 ends this paper.

## 2 Motivation and Inputs

The present work is principally motivated by the conviction that the notion of "volume" has an existence independent from any metric – in fact, it must be considered to be a *pre-metric* concept. Curiously, such an independent volume had not been taken into consideration in physical theories until relatively recently. Even in differential geometry it is almost ignored. Perhaps the reason for this neglect is that in most circumstances there is a *canonical* volume element, based on other geometric structures considered to be more basic. For example, in Riemannian geometry the volume element density is defined in terms of the metric. In particular, the important operation of *Hodge dual* for differential forms is conventionally based on such a Riemannian volume element.

However, from the gravitational sector of the low-energy limit of string theory (i.e., compactification to dimension  $n = 4$ ) there comes the suggestive hint (cmp. Garfinkle, Horowitz and Strominger [1]) that when both a dilaton scalar and a two-form are present, the dilaton scalar enters the expression for the volume element density when defining a "natural" Hodge dual operator.<sup>1</sup> This was taken as the starting point to develop a theory of *geometric dilaton gravity* (Graf [4]). Although the particular coupling does not exactly correspond to the coupling suggested by string theory, wormholelike solutions were obtained.

Recently a breakthrough on Ricci flow methods was achieved by Perelman [5-7], developing the decisive tools to solve the famous Poincaré conjecture on the topological characterization of the three-sphere. Based on 3-d (compact and positive-definite) Riemann spaces, smoothly deformed by a *Ricci flow* (RF), the "basic" RF equation was originally defined by

$$\partial_t g_{ik} = -2 R_{ik}, \tag{1}$$

where  $R_{ik}$  is the Ricci-tensor corresponding to a "time-dependent" three-metric  $g_{ik}$ . Also a special class of diffeomorphisms was considered, with vector  $\vec{v}$  which is essentially the gradient of a scalar  $\phi$  in the sense of  $v^i = g^{ik} \partial_k \phi$ . The so generalized RF equation then becomes

$$\partial_t g_{ik} = -2 (R_{ik} + \nabla_i \nabla_k \phi). \quad (2)$$

Although such equations have already been studied since the early eighties starting with the seminal works of Hamilton [8] and DeTurck [9], an essential insight of Perelman was to recognize that the r.h.s. of this equation<sup>2</sup> can be expressed as the *gradient* of an appropriate functional. This functional involves a "measure" given in terms of an *independent volume element*. The gradient property allows to apply a series of standard analytical tools. And the introduction of the measure gives an extra flexibility, analogous to a choice of gauge.<sup>3</sup>

Since the works of Hamilton and DeTurck just mentioned, geometric flows have been applied to a variety of geometric, topological and analytical problems.<sup>4</sup> Flow-like equations are also not unfamiliar to physicists, the earliest and most well-known being the renormalization group equations in quantum field theory (for an introduction, see Mitter [14]), and also the nonlinear  $\sigma$ -model (e.g., Lott [15], Oliynyk, Suneeta and Woolgar [16], Tseytlin [17]). After Ellis [18] called attention to the cosmological "fitting problem", the usefulness of the Ricci flow to deal with volume-averaged inhomogeneities was immediately recognized and continues to be an active area of research (e.g., Carfora and Marzuoli [19], Buchert and Carfora [20], and the recent review by Buchert [21]). An overview of flow techniques in physics is given in Bakas [22].

Another motivation came however with the insight, that the basic equations derived from the low-energy limit of string theory can be put into a form suprisingly similar to Ricci flow equations when besides the metric only a dilaton scalar is kept. The main formal difference is the number of dimensions and the signature of the corresponding Riemann spaces: whereas the "classic" RF equations refer to a parameter-dependent truly Riemannian three space evolved by an extrinsic "time" parameter, the reformulated string theory equations refer to a four-dimensional Lorentzian spacetime, which is evolved along the directions of an intrinsic vector field.

Neither the "classic" RF approach nor string theory suggest any hints about the coupling of geometry to external matter fields. Therefore we will spend some time to prepare the field in order to include other external matter. As not only geodesy of the motion of "test particles" will in general be violated, but also "conservation" (in the sense of  $\nabla \cdot T = 0$ ), we will be especially careful to lay a coherent and stringent foundation. The Noether identities will be our main guide. As result we will get Ricci flow gravity (RFG).

For the history of scalar-tensor theories and their current status, we refer to Brans [23], and to the recent monographs of Fujii and Maeda [24] and Faraoni [25].

### 3 Volumetrical Manifolds

In Graf [4] we already introduced the notion of a volume manifold and its specialization for the case a nondegenerate metric exists. Let us briefly recapitulate the main notions. First, we intro-

duced the fundamental concept of a *volume structure*, which has to be considered as independent from any metric. This is just a non-negative  $n$ -form density  $\omega$ , and makes the manifold a *volume manifold*. Secondly, we will need of course a *metric structure*. However, it does not need to be compatible with the volume structure. This difference is encoded by means of the *volume scalar*  $\phi$  by  $\omega = \omega e^{-\phi}$ ,<sup>5</sup> where  $\omega := |\det g|^{1/2} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$  is the usual metrical (i.e., Riemannian) volume element density. Furthermore, for the metric derivative along a vector  $X$  we have  $\nabla_X \omega = -(X \cdot \partial\phi)\omega$  (the dot denoting a contraction) as a measure of incompatibility.

Such a manifold, endowed both with an independent volume and a metric structure, we will denote by **volumetrical manifold**. Whereas the manifold is considered to be smooth, both metric and volume element density are allowed to diverge or to be degenerate, when they are not locally smooth.

Already in a volume manifold the Gauss theorem for a vector  $\xi$  can be expressed very compactly in terms of differential form densities as

$$\int_{b\Omega} \xi \cdot \omega = \int_{\Omega} d(\xi \cdot \omega), \tag{3}$$

where  $b\Omega$  is a two-sided hypersurface bounding the  $n$ -dimensional region  $\Omega$ . The scalar factor  $\text{div } \xi$  in the relation  $d(\xi \cdot \omega) = (\text{div } \xi) \omega$  is also better known under the name of *divergence* of the vector  $\xi$ . Evidently the div-operator only depends on the particular choice of  $\omega$  and not on any metric.

#### 4 Volume Preserving Lie Flows

In a differentiable manifold, the thing coming closest to an autonomous first order differential equation for a "vector"  $x(t)$ ,

$$\dot{x} = f(x), \tag{4}$$

is the notion of a *Lie equation*

$$\mathcal{L}_{\xi} F = G, \tag{5}$$

where  $F$  and  $G$  are geometric objects (e.g., tensors),  $\xi$  is some vector field and  $\mathcal{L}_{\xi} F$  denotes the Lie derivative of  $F$  along  $\xi$ . In the simplest case  $\xi$  and  $G$  are considered as given and  $F$  to be determined. However in the applications we have in mind, *all* elements of the equation will be dynamically determined,  $G$  depending nonlinearly on  $F$  and its partial derivatives, and even  $\xi$  will become dynamical.

In the theory of ordinary differential equations, such systems of first-order equations which guarantee the long-term existence both to the future and the past, are also called flows and can be characterized by the *one-parameter Abelian group property* of their solutions. As well-known, Lie operators share exactly the same one-parameter Abelian group property (at least locally) by means of the exponential map. We can therefore speak of a *Lie flow*.

In a volume manifold, a Lie flow with vector  $\xi$  is called *volume-preserving* (or VP)<sup>6</sup> if

$$\mathcal{L}_\xi \omega = 0, \text{ or equivalently, } \operatorname{div} \xi = 0. \tag{6}$$

In the rest of this paper we will try to make plausible a particular scalar-tensor extension of Einstein gravity in terms of a *volume-preserving Ricci Lie* flow in a volumetric manifold.

### 5 A choice of Scalar-Tensor Field Equations

Assuming that the total Lagrangian (or at least the field equations) can be uniquely split into a pure geometrical part and the "physical" part, we can already draw important conclusions about both the algebraic and the differential properties of the "physical" energy-momentum tensor just from examining the purely geometrical part. Note that whereas Perelman's analysis is "metric-centered", with an auxiliary scalar, in the following physical applications this scalar will play a role at the same conceptual level as the metric. Therefore the "classical" Lagrangian approach is appropriate.

Let us start with the "geometrical" Lagrangian living on a volumetric manifold  $M$ ,

$$\mathcal{L} = \omega (R + \lambda (\nabla \phi)^2), \tag{7}$$

where  $\omega := e^{-\phi} \omega$  and  $(\nabla \phi)^2 := g^{ij} \partial_i \phi \partial_j \phi$  and  $\lambda$  is a constant parameter. Despite its simple form it not only includes the one used initially by Perelman and in the low-energy limit of string theory (when ignoring the axion and the other moduli fields), but which also is essentially the Jordan-Brans-Dicke Lagrangian.

Defining the *volume factor*  $\Phi := e^{-\phi}$ , we then have as variational derivatives (up to volume element,  $g$ -dualizations of  $P$  and a common sign -1)

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta g_{ik}} \sim P_{ik} &:= G_{ik} - \Phi^{-1} (\nabla_i \nabla_k - g_{ik} \Delta) \Phi \\ &+ \lambda \Phi^{-2} (\nabla_i \Phi \nabla_k \Phi - \frac{1}{2} g_{ik} (\nabla \Phi)^2), \end{aligned} \tag{8}$$

$$\frac{\delta \mathcal{L}}{\delta \phi} \sim Q := R - 2\lambda \Phi^{-1} \Delta \Phi + \lambda \Phi^{-2} (\nabla \Phi)^2, \tag{9}$$

where  $G$  denotes the Einstein tensor  $G_{ik} := R_{ik} - \frac{1}{2} R g_{ik}$  and  $\Delta := \nabla^2$  the d'Alembertian. For the above Lagrangian the *Noether identity* can be written compactly as

$$\text{div} (\tilde{P}_k^i \xi^k) = P^{ik} \xi_\xi g_{ik} + Q \xi_\xi \phi, \tag{10}$$

with some tensor  $\tilde{P}_k^i$  to be determined by it. More conventionally,

$$\nabla_i (\Phi \tilde{P}_k^i \xi^k) = \Phi (P^{ik} \xi_\xi g_{ik} + Q \xi_\xi \phi). \tag{11}$$

As this identity must hold for any smooth vector  $\xi$ , we get separately

$$\tilde{P}_{ik} = 2P_{ik} \quad \text{and} \quad \nabla_i (\Phi \tilde{P}_k^i) = Q \partial_k \Phi. \tag{12}$$

Note that from equation (11) follows conservation in the proper sense, if  $\bar{\xi}$  is a *simultaneous* Killing vector both of the metric and of the scalar, even if  $Q \neq 0$ .

The following algebraico-differential relations evidently hold:

**symmetry:**  $\tilde{P}_{ik} = \tilde{P}_{ki}$ , and

**balance:**  $\nabla_i (\Phi \tilde{P}_k^i) = Q \partial_k \Phi$ .

Up to this point we only made use of identities, but not of any field equations. In particular if we equate  $P$  and  $Q$  to their corresponding physical quantities, these algebraico-differential relations will be "impressed" on them. In fact, it is not even necessary that they follow from a Lagrangian.

But note that there is a dependency not only on  $\lambda$  but also on the number  $n$  of dimensions, making  $n = 4$  and  $n = 3$  (for  $\lambda = 1$ ) somewhat special. This is most evident in the relation

$$P + Q = ((n - 1) - 2\lambda) \Phi^{-1} \Delta \Phi - \frac{\lambda}{2} (n - 4) \Phi^{-2} (\nabla \Phi)^2, \tag{13}$$

where  $P$  is the "trace"  $P := g^{ik} P_{ik}$ . Assuming from now on  $n = 4$ , this simplifies to

$$P + Q = (3 - 2\lambda)\Phi^{-1} \Delta\Phi. \tag{14}$$

Let us define the *geometrical energy-momentum tensor*  $P_{ik} := 1/2 \tilde{P}_{ik}$  and more closely examine the corresponding balance relation

$$\nabla_i(\Phi P_k^i) = \frac{1}{2} Q \partial_k \Phi. \tag{15}$$

The following cases can be distinguished, when equating the geometrical quantities  $P$  and  $Q$  to their "physical" counterparts  $P_m$  and  $Q_m$ :

- a) **Pure Einstein**,  $\phi = 0$
- b) **Conformally Einstein**,  $\lambda = 3/2$
- c) **"Conservative"**,  $Q = 0$
- d) **VP flow**,  $\Delta\Phi = 0$
- e) **Fully dynamical**.

Each of these choices will now be discussed individually.

### 5.1 Pure Einstein

This is just the "compatibility mode", or "Einstein-limit"  $\phi \rightarrow 0$  (if it exists). It is thus a volumetric theory only in the trivial sense of  $\phi = 0$ .

### 5.2 Conformally Einstein

The system of equations is underdetermined. This becomes evident by going to the Einstein-frame by means of the conformal transformation  $g'_{ik} = e^\phi g_{ik}$ , where the scalar field drops out completely.

### 5.3 "Conservative": Jordan-Brans-Dicke

From  $Q = 0$  there follows "conservation" in the usual sense of  $\nabla_i(\Phi P_k^i) = 0$ . This assumes the particular relation  $P\Phi = (3 - 2\lambda) \Delta\Phi$ , which could either be postulated or obtained by a specially tailored Lagrangian. In order to have a more familiar looking equation,  $P_{ik}$  could be equated to the physical quantity  $T_{ik}$  over  $P_{ik} = \Phi^{-1} T_{ik}$ , so that in fact "conservation" in the sense of  $\nabla_i T_k^i = 0$  would result. This kind of "conservation" was considered as absolutely

essential in the closely related scalar-tensor theories of Jordan and Brans-Dicke (in short, JBD theories).<sup>7</sup> In fact, this can be achieved as follows: their scalars  $\phi$  (resp.  $\kappa$ ) must be identified with  $\Phi$ , and we must identify their coupling parameters  $-\omega$  (resp.  $\zeta$ ) with  $\lambda$ . Moreover  $\lambda \neq 3/2$  has to be assumed, otherwise the conformally Einstein theory would result. Then  $1/\phi$  (resp.  $\kappa$ ) is interpreted as the (variable) gravitational constant. Both for Jordan's and Brans-Dicke's material energy-momentum tensor it is supposed that it does *not* depend on the scalar  $\phi$ . However in Jordan's theory it is the product  $\kappa^2 T_k^i$  which is "conserved".<sup>8</sup> In particular, for a "dust model" geodesy of  $\bar{u}$  (resp.  $\kappa^2 \bar{u}$ ) and conservation of  $\rho \bar{u}$  (resp.  $\kappa^2 \rho \bar{u}$ ) still follow when staying in the original conformal ("Jordan") frame.

### 5.4 Volume-preserving Flow

When not a conformally Einstein coupling, from equation (14) the condition  $P + Q = 0$  is equivalent to  $\Delta \Phi = 0$ , which in turn is equivalent to volume-preservation  $\mathcal{L}_\xi \omega = 0$ . This translates to the scalar condition

$$P_m + Q_m = 0 \tag{16}$$

for the corresponding "material" quantities. Let us call such a coupling to matter a *volume-preserving material coupling* (VPMC), and assume it to hold throughout this section. Then

$$\nabla_i(\Phi P_k^i) = -\frac{1}{2} P \partial_k \Phi. \tag{17}$$

Evidently, if the trace  $P_m$  of the energy-momentum tensor vanishes the VPMC is satisfied if we set  $Q_m = 0$ . Then the standard "conservation" continues to hold. This is the case e.g. for the Maxwell field.

As an important example where  $P \neq 0$ , let us take the ideal fluid model, where the material 6 energy-momentum tensor is given by  $P_m^{ik} := T^{ik} = \rho u^i u^k + p \Pi^{ik}$ , and  $\Pi_k^i := \delta_k^i + u^i u_k$  is the projector orthogonal to the trajectory with (normalized) tangent  $\bar{u}$ . Its trace is  $T = 3p - \rho$ . To satisfy the VPMC, we must set  $Q_m = -T$ . Specializing to pure dust we get  $\nabla_i(\Phi \rho u^i u^k) = \frac{1}{2} \rho g^{ik} \partial_i \Phi$ . Splitting into tangential and orthogonal parts, we then get the separate equations

$$\nabla_i(\rho u^i) = -\frac{1}{2} \rho \dot{\phi} \quad \text{and} \quad \dot{u}^i = \frac{1}{2} \Pi^{ik} \partial_k \phi. \tag{18a+b}$$

Due to the nonvanishing of the r.h.s. of these equations, both "conservation of matter" and geodesy for "test particles" are broken unless  $\phi = \text{const}$ . And due to the particular form of the



equation of motion 18b (i.e., being proportional to a gradient) we have in fact got a *Newton-Nordström-term*.<sup>9</sup>

Concerning the divergence expression (18a), it can nevertheless be rewritten as a *conservation law*,  $\nabla_i(\Phi^{1/2}\rho u^i) = 0$ . Therefore, for such a theory with volume-preserving flow, both the equation of motion as well as the "conservation of dust matter" are not anymore the well-known standard expressions from Einstein or Jordan-Brans-Dicke theory. It can be expected that this will have profound consequences in a cosmological setting.

We will continue the discussion of volume-preserving theories in section 6, where we further specialize to the coupling parameter  $\lambda = 1$ .

### 5.5 Fully dynamic Scalar Field

Here the scalar  $\phi$  is dynamically determined by a set of field equations obtained via a suitable Lagrangian, and no case of the previously discussed ones fits. This would normally be the "standard" procedure in physics, where not only the Lagrangian is set up as a linear combination of individual Lagrangians, each one describing a different matter model, but in addition possibly introducing some extra "potential terms" containing  $\phi$  and  $\partial\phi$ , or even let  $\lambda$  depend on  $\phi$ . However this will in general prevent a simple geometrical interpretation in terms of a flow, and in particular will lack the crucial VP property. For example, our geometric dilaton gravity (Graf [4]) belongs to this more general class.

## 6 Ricci Flow Gravity

The class of volume-preserving volumetric theories can be further refined by requiring the particular value  $\lambda = 1$  of the coupling, as is common in the low-energy limit of string theory. With this particular value the field equations can be rearranged into an explicit flow-like form and we get the **Ricci flow gravity equations** (RFG equations)

$$\mathcal{L}_\xi g_{ik} = 2(R_{ik} - \bar{T}_{ik}), \tag{19}$$

$$\mathcal{L}_\xi \mathfrak{O} = 0, \tag{20}$$

describing **Ricci flow gravity** (RF gravity). Here the *flow vector* is defined in terms of the volume scalar as  $\vec{\xi} = -g^{-1}\partial\phi$ ,<sup>10</sup> and  $\bar{T}_{ik} := 8\pi(T_{ik} - \frac{1}{2}Tg_{ik})$ .<sup>11</sup> In contrast to the JBD equations, they have a much simpler structure and an immediate *geometric* character. Through their particular flow-like form, they exhibit a strong *dynamical* touch: broadly speaking, the rate of change of the metric is driven by the difference of the geometrical and the physical energy momentum tensors. Evidently, when the flow vector can be ignored (e.g., when it vanishes) equations equivalent to Einstein's are obtained. In this sense Einstein gravity is a special case of Ricci flow gravity.

The RFG vacuum equations are equivalent to JBD's vacuum equations with  $\omega = -1$ . More remarkably is the fact that they are also equivalent to the equations following from the low-energy limit of string theory when the standard dilaton coupling with  $\lambda = 1$  is chosen and besides the metric only the dilaton scalar is kept. And of course there is a strong resemblance to Perelman's Ricci flow equations which can be made more evident as follows. Consider tentatively on  $M := M_3 \times T$  the vector  $\vec{\xi} := -(\partial_t + \vec{v})$  and the metric  $g_{ik}$  with line element  $ds^2 = d\sigma^2 - dt^2$ . Then the generalized RF equation (2) can be written as  $\mathcal{L}_{\vec{\xi}} g_{ik} = 2R_{ik}^{(3)}$ , which differs (in content, but not in form) only on the r.h.s. from the corresponding RFG vacuum equation.<sup>12</sup>

This coincidence of seemingly different approaches could signal a deeper *raison d'être*.

### 7 On the physical interpretation of the scalar $\Phi$

The scalar  $\Phi$  was here interpreted geometrically in the context of a volumetrical manifold as the volume factor. In the theories of JBD the corresponding scalar is essentially interpreted as "gravitational constant"  $\kappa$  – more precisely  $\kappa = \Phi$  in Jordan's theory, whereas  $\kappa = 1/\Phi$  in Brans-Dicke theory. However this physical interpretation cannot be upheld anymore in a volume-preserving theory like RFG where  $\kappa$  is constant.

Due to the fact that the volume factor  $\Phi$  of a volume-preserving theory obeys the d'Alembertian wave equation  $\Delta\Phi = 0$  it must therefore be interpreted as a *massless real scalar field*. By the tenets of relativistic quantum mechanics this corresponds to a *totally uncharged massless bosonic particle*.<sup>13</sup> The volume-preservation will also be instrumental to guarantee an almost perfect compliance with the standard solar system tests of gravity. This is a fair return for the price we had to pay for giving up the geodesy of "test particles".

### 8 Compliance of Ricci flow gravity with the Standard Tests

For the standard solar system tests the corresponding generalization of the Schwarzschild metric is needed. The general asymptotically flat and static spherically symmetric vacuum RFG solution with  $\Phi \rightarrow 1$  for  $r \rightarrow \infty$  can be written as

$$ds^2 = -Y^{\gamma-\sigma} dt^2 + Y^{-\gamma-\sigma} (dr^2 + Z^2 d\Omega^2), \tag{21}$$

$$\begin{aligned} \Phi &\equiv e^{-\phi} = Y^\sigma, \quad \text{where} \\ Y &:= \frac{r-r_+}{r-r_-}, \quad Z^2 := (r-r_+)(r-r_-), \end{aligned} \tag{22}$$

with  $\gamma^2 + \sigma^2 = 1$ , and it is assumed that  $r \geq r_+ > r_- \geq 0$ .<sup>14</sup> Being for  $r > r_+$  a vacuum RFG solution, it is also the corresponding general JBD vacuum solution in the "Jordan"-frame. But whereas in JBD gravity the source of the volume factor  $\Phi$  for a mass point has to be a certain nonzero distribution supported by  $r = r_+$ , in RF gravity due to  $\Delta\Phi = 0$  it must be sourceless. This can be shown to hold even for a compactly supported smooth static spherically symmetric energy-momentum tensor as source, if both metric and volume factor are smooth and the manifold is simply-connected. Therefore for RFG  $\sigma = 0$ , whereas for JBD  $\sigma = 1/2 (3 + 2\omega)^{-1}$ .

This can also be expressed more conveniently in terms of the "dilaton charge"  $D$ , which in the context of the low-energy limit of string theory is defined for a static solution with Killing vector  $\eta$  (normalized to  $\eta^2 = -1$  at infinity) as

$$D = \frac{1}{4\pi} \oint \eta \cdot \xi \cdot \omega, \tag{23}$$

where the integral is taken over a closed and externally orientable 2-sphere at spatial infinity.<sup>15</sup> For RF gravity the two-form density  $\chi := \eta \cdot \xi \cdot \omega$  is even closed,  $d\chi = 0$ , for any stationary solution with Killing vector  $\eta$  so that the above integral only depends on the homology class of the closed externally orientable 2-sphere. In particular it vanishes if this 2-sphere bounds. With the flow vector  $\vec{\xi} = \sigma (r_+ - r_-) Y^{\gamma+\sigma} Z^{-2} \partial_r$  for the above solution this results in  $D = \sigma (r_+ - r_-)$ . For vanishing dilaton charge the Schwarzschild solution is evidently reobtained after substituting  $r$  by  $r_+ + r_-$ , setting  $m = (r_+ - r_-)/2$  and assuming  $m > 0$ . Thus for the standard solar-system tests the flow vector vanishes and we have *full compatibility* with Einstein gravity, which passes these tests with ever increasing accuracy (cf. Will [32]).<sup>16</sup>

Of course where the flow vector does not vanish, Ricci flow gravity and Einstein gravity will lead to different answers. Using heuristically the term "charge" as introduced above (possibly without stationarity) we note that differently from the "mass charge"  $m$ , the "dilaton charge"  $D$  can have any sign.<sup>17</sup> Therefore the contributions to the total charge of a collection of "charged regions" can still sum up to zero, so as to make the Newton-Nordström-terms of the equation of motion insignificant for sufficiently big distances.<sup>18</sup> This should be considered to be in fact the case for "ordinary matter" building up planets, stars and perhaps, galaxies. Significant differences are however to be expected in a cosmological setting, where the "big bang" will affect the behaviour of the volume scalar  $\phi$ .

Although for the "compliant mode"  $\phi = \text{const}$  evidently it makes no difference if the metric is interpreted in the geometric frame or in the Einstein frame, this is not so in the general case where even the equations of motion for a point particle are modified. We have to chose the *particular*

*conformal frame*, where the field equations find their "most natural expression". This is the *geometrical* frame with an independent volume element density.

## 9 Conclusion

Motivated by the neglect of the notion of an independent volume and led by the appeal of Perelman's approach to solve the Poincaré conjecture, as well as by the equations following from the low-energy limit of string theory, we developed the equations of Ricci flow gravity as a natural extension of Einstein gravity. The main differences with regard to other scalar-tensor theories were worked out in the framework of volumetric manifolds. The volume-preservation of the flow turned out to be of decisive importance for the theory and allowed it to essentially agree with Einstein's under non-cosmological settings and not too small distances in the case of vanishing "total dilatonic charge".

## Notes

<sup>1</sup>for the dilaton general concept, see Sundrum [2]; for the connection of gravity to strings, see Ortín [3]

<sup>2</sup>the expression in parenthesis in the r.h.s. of equation (2) has its own measure-theoretic meaning and is also known under the name "Bakry-Émery" tensor (cmp. Lott [10])

<sup>3</sup>in particular he envisages *volume-preserving flows* and certain diffeomorphic images thereof

<sup>4</sup>see the recent monograph of Chow and Knopf [11] on Ricci flows (not covering Perelman's contributions), the introduction by Topping [12], and the lecture notes by Morgan and Tian [13]

<sup>5</sup>the factor -1 of  $\phi$  is purely conventional – here we follow Perelman [5], in contrast to string theory, where -2 is preferred

<sup>6</sup>this is a local concept in contrast to the much weaker global definition of Huisken [26]

<sup>7</sup>cmp. Jordan [27], Weinberg [28], part II, ch. 7, §3 and Fujii and Maeda [24]

<sup>8</sup>this is suggested by his interpretation of the Kaluza-Klein decomposition

<sup>9</sup>recall that around 1912–13 Nordström developed a precursor relativistic gravitational theory, where the gravitational potential  $\phi$  obeys a Minkowskian potential equation,  $\Delta \phi = 0$ . This was shown in 1914 by Einstein and Fokker to admit a conformally Minkowskian formulation

<sup>10</sup>the arbitrary minus-sign is taken in view of cosmological applications

<sup>11</sup>we use throughout the sign- and units conventions of Misner, Thorne and Wheeler [29]

<sup>12</sup>the connection between some solutions of the Ricci flow equations for  $n = 3$  and solutions of the Einstein equations for  $n = 4$  was further elaborated by Bleecker [30] and by List [31]

<sup>13</sup>except for a "dilaton charge"; see the discussion in next section

<sup>14</sup>in the "degenerate" case  $r_+ = r_-$  the metric is locally flat and the volume factor constant

<sup>15</sup>cmp. Garfinkle, Horowitz and Strominger [1]

<sup>16</sup>to compare, for JBD gravity to pass the current tests  $|\omega| > 4 \times 10^4$  must be assumed

<sup>17</sup>this allows the dilaton scalar to act "repulsively", as shown in Graf [4]

<sup>18</sup>e.g., for a "multipole charge" when the distance is much bigger than the individual "charges"

### Acknowledgements

I thank the members of the relativity group of the University of Vienna for discussions and useful suggestions, and in particular Peter C. Aichelburg. Partial financial support from the Fundación Federico is also acknowledged.

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